Linear Programming: Lecture 21.

(1)

Integrality & duality.

Recall: if $P = \{x : Ax \leq b\}$ is a polytope, then there is a basic feasible soln (bfs) optimal for minimizing $c^T x$.

As always, $A \in \mathbb{R}^{m \times n}$

we said: algorithm to obtain all basic (feas) solutions:

- choose $I \subseteq [m]$, set of n linearly independent rows of A
- solve ~~x_I~~ $x = A_I^{-1} b_I$, check if feasible
- among all such solutions, pick optimal

LPs give fractional solutions. When is solution integral?
(i.e., each coordinate of x is an integer).

Sufficient condition: all bfs are integral

$\Rightarrow \forall I \subseteq [m]$, $A_I^{-1} b_I$ is integral

let ~~$x_j =$~~ $x = A_I^{-1} b_I$

then j^{th} component $x_j = \dfrac{|A_I^j|}{|A|}$ (Cramer's Rule)

where ~~A~~ A_I^j is A_I w/ j^{th} column replaced by b_I.

thus, if both $|A_I^j|$, $|A|$ are integral, then all basic solutions are integral

Total unimodularity :

Matrix ~~$A \in \mathbb{R}$~~ $A \in \{0,1,-1\}^{m \times n}$ is TU iff every square submatrix of A has determinant $\in \{+1, 0, -1\}$.

~~(clearly this is sufficient, but not necessary)~~

Theorem : Let A be TUM, $b \in \mathbb{Z}^n$. Then $P = \{x : Ax \geq b\}$ is integral, i.e., every bfs is integral.

(proof easy).

We said earlier can find ~~a b~~ an optimal bfs for an LP in ~~linear~~ polynomial time. If polytope is integral, this means we can find an optimal integral solution in poly time. This is a very powerful tool....

~~[Claim: The following statements are equivalent]~~

Claim : ~~If~~ A is TU (~~the same~~) iff the following are TU:

(i) $-A$

(ii) A^T (proof easy)

(iii) $[A \ e_i]$, $[A \ -e_i]$

(iv) $[A \ I]$, $[A \ -I]$

(v) $[A \ A^i]$, i.e., a column is repeated, and $[A \ -A^i]$

Corollary : If A is TU, and a, b, c, d are integral vectors, then the polytope $Q = \{x : a \geq Ax \geq b, \ c \geq x \geq d\}$ is integral.

So how do we determine if A is TU? One way is to check all square submatrices. But we can give some sufficient (& necessary!) conditions for this also.

Theorem (*): Let $A \in \{+1, 0, -1\}^{m \times n}$. Then A is TUM iff each set S of columns can be divided into sets S_1, S_2 s.t.

$$\sum_{i \in S_1} A^i - \sum_{i \in S_2} A^i \in \{-1, 0, 1\}^m$$

(proof skipped)

Here's a simple matrix that is NOT TU:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$ since ~~consider~~ $|A| = -2$.

(Can also see that $Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ has soln $x = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$)

ok, so now lets see some interesting LPs with integral polytopes (actually TU matrices for constraints)

Recall our LP for maximum bipartite matching:

$$\max \sum_e x_e$$

$$\forall v: \sum_{e \text{ incident to } v} x_e \leq 1$$

$$\forall e: \quad x_e \geq 0$$

We want to show that the constraint matrix is TU

Lets take a closer look:

$$
A = \begin{array}{cc}
 & |E| \\
\begin{array}{c} |V| \\ \\ \\ |E| \end{array} &
\left[\begin{array}{cccccc}
1 & 1 & 1 & 0 & \cdots & 0 \\
0 & 1 & 0 & 1 & \cdots & 1 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
\vdots & & & & & \\
\hline
1 & 0 & 0 & \cdots & & 0 \\
0 & 1 & 0 & \cdots & & 0 \\
 & & \vdots & & & \\
0 & 0 & 0 & \cdots & & 1
\end{array}\right]
\end{array}
= \begin{bmatrix} A' \\ I \end{bmatrix}
$$

we will show that A'^T is TU

again: let's write down A'^T

$$
\begin{array}{cc}
 & |V| \\
|E| &
\left[\begin{array}{c|c}
1 \; 0 \cdots & 0 \; 1 \; 0 \cdots \\
1 \; 0 \cdots & 1 \; 0 \; 0 \cdots \\
 & \\
 & \\
\end{array}\right] \\
 & \quad A \qquad\qquad B
\end{array}
$$

Since graph is bipatite, $V = A \cup B$ all edges b/w A & B.

+ each row has two non-zero entries, other as zero.

~~[illegible]~~ g

Now we will show A'^T is TU.

Given any $S \subseteq V$ (colums), let $S_1 = S \cap A$, $S_2 = S \cap B$.

Then consider $\underbrace{\sum_{i \in S_1} (A'^T)^i}_{\in \{0,1\}^m} - \underbrace{\sum_{i \in S_2} (A'^T)^i}_{\in \{0,1\}^m}$ ~~$\in \mathbb{Z}$ (sing)~~

diff $\in \{1, 0, -1\}^m$.

Thus A'^T is TU $\Rightarrow$ A' is TU $\Rightarrow$ $\begin{bmatrix} A' \\ I \end{bmatrix}$ is TU $\Rightarrow$ A is TU.

Thus by solving the linear program, can find maximum ~~card~~ bipartite matching in poly-time.

Further, can find max-wt. bipartite matching also:

$$\max \ (w^T x) \ \sum_e w_e x_e$$

$$\text{s.t.} \quad \forall v, \ \sum_{e \text{ incident to } v} x_e \leq 1$$

$$\forall e, \quad x_e \geq 0.$$

(note that w ~~x~~ may not be integral).

Example 2: Maximum flows. (why are we looking for integral soln. here?)

Recall LP:

$$\max \sum_{e \text{ out of } s} x_e$$

$$\text{s.t.} \quad \forall v \neq s,t \quad \sum_{e \text{ into } v} x_e - \sum_{e \text{ out of } v} x_e = 0$$

$$\forall e \quad x_e \leq c_e$$

$$\forall e \quad x_e \geq 0$$

constraint matrix:

$$A = \begin{array}{c} \\ |V|-2 \\ \\ \\ \\ |E| \\ \\ |E| \end{array} \begin{array}{c} |E| \\ \left[\begin{array}{ccccc} 1 & 1 & -1 & 0 & \\ -1 & 0 & 0 & 0 & \\ 0 & -1 & 0 & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 1 & 1 & \\ \hline & & I & & \\ \hline & & I & & \end{array}\right] \end{array} \ A'$$

~~will show~~ A'^T is TU: for any S, let $S_1 = S$, $S_2 = \phi$.

Duality.

Consider the LP:

$$\min \quad x_1 + 2x_2$$

$$x_1 - x_2 \geq 3 \quad \leftarrow y_1$$

$$2x_1 + x_2 \geq 1 \quad \leftarrow y_2$$

$$x_1, x_2 \geq 0$$

suppose ~~now~~ we want a lower bound on the optimal solution.

Let's multiply the constraints by $y_1, y_2 \geq 0$, and add.

then: $x_1(y_1 + 2y_2) + x_2(-y_1 + y_2) \geq 3y_1 + y_2$

if we further constrain y_1, y_2 s.t. $y_1 + 2y_2 \leq 1$

$-y_1 + y_2 \leq 2$

then LHS is a lower bound (less than) on $x_1 + 2x_2$

hence, RHS is a lower bound on $x_1 + 2x_2$

i.e., $3y_1 + y_2$ is a lower bound on $x_1 + 2x_2$

s.t. $y_1 + 2y_2 \leq 1$ $\qquad$ $x_1 - x_2 \geq 3$

$-y_1 + y_2 \leq 2$ $\qquad$ $2x_1 + x_2 \geq 1$

$y_1, y_2 \geq 0$ $\qquad$ $x_1, x_2 \geq 0$

hence, $\max 3y_1 + y_2$ is a lower bound on $\min x_1 + 2x_2$

this is called the dual LP $\rightarrow$

$y_1 + 2y_2 \leq 1$ $\qquad$ $x_1 - x_2 \geq 3$

$-y_1 + y_2 \leq 2$ $\qquad$ $2x_1 + x_2 \geq 1$

$y_1, y_2 \geq 0$ $\qquad$ $x_1, x_2 \geq 0$

this is the primal

For every LP, there is a "dual" LP that provides a lower bound on the optimal value (assuming minimization objective).

(what if "primal" LP has unbounded optimal value? dual LP is infeasible).

Further, dual of the dual is the primal.

more generally:

Primal: $\min \sum_i c_i x_i$		Dual: $\max \sum_j b_j y_j$
$a_{11}x_1 + \dots + a_{1n}x_n \geq b_1$	$\times y_1$	$a_{11}y_1 + \dots + a_{m1}y_m \leq c_1$
$a_{21}x_1 + \dots + a_{2n}x_n \geq b_2$	$\times y_2$	$a_{12}y_1 + \dots + a_{m2}y_m \leq c_2$
$\vdots$	$\vdots$	$\vdots$
$a_{m1}x_1 + \dots + a_{mn}x_n \geq b_m$	$\times y_m$	$a_{1n}y_1 + \dots + a_{mn}y_m \leq c_n$
$x_1, \dots, x_n \geq 0$		$y_1, \dots, y_m \geq 0$

or:

$$\begin{array}{ccc} \min\ c^T x & \xleftrightarrow{\text{dual}} & \max\ b^T y \\ Ax \geq b & & A^T y \leq c \\ x \geq 0 & & y \geq 0 \end{array}$$

~~any feasi~~ x

and if x, y feasible for primal & dual, then $c^T x \geq b^T y$.

This is called <u>weak duality</u>.